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Event–phase-space structure: an alternative to quantum logic

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Abstract. The main aim of this paper is to examine two new possibilities in the axiomatic foundations of quantum mechanics: first, the possibility of introducing a non-symmetric transition probability between pure states, and second, showing that the concept of orthocomplementation in the logic of events is unnecessary and of secondary importance. Presented here is an axiomatic scheme, which does not involve the concept of orthocomplementation and yet has all the advantages of the well-known quantum logic axiomatics, because our generalised logic of events admits an extension, which is a complete orthocomplemented orthomodular lattice with the covering law holding in it. Thus, the approach to quantum axiomatics presented here may be seen as answering both the old questions of the quantum logic approach (e.g. the questions of the complete lattice structure of the logic, atomicity, the validity of the covering law) and the question concerning the necessity of the orthocomplementation in the logic of events, recently raised by Mielnik.

1. Introduction

The purpose of this paper is to indicate two new possibilities in the axiomatic foundations of quantum theory: first, the possibility of introducing a non-symmetric transition probability between pure states (see also Guz 1979), and second, showing that the concept of orthocomplementation in the propositional logic is unnecessary. This confirms the recent claim (Mielnik 1976) that the orthocomplementation, having in fact no physical justification, is of secondary importance for quantum axiomatics.

The postulates which we assume here are, in fact, implied by those of the well-known quantum logic approach (Guz 1978a). Later on (§ 5) all these axioms are reformulated in terms of the transition probability and the filters (operations) acting on the set of pure states of the physical system under study. This leads us to the concepts of transition probability space (see also Guz 1979, where this notion has been introduced) and filter, the latter being defined as an operation transforming the former into itself.

Presented here is a system of axioms possessing all the advantages of the recently formulated one (Guz 1978b), but without any reference to orthocomplementation. Our generalised propositional logic (called here the logic of events) admits an extension, a complete orthocomplemented orthomodular lattice in which the covering law holds (§ 6), and therefore the well-known representation theorem (e.g. Piron 1964, MacLaren 1964, Varadarajan 1968, Maeda and Maeda 1970) holds for it, provided we assume the extended logic to be irreducible and of projective dimension not smaller than four. Therefore, the approach to quantum axiomatics presented here may be seen as answering both the old questions of the quantum logic axiomatic framework (e.g. the questions of the complete lattice structure of the logic, atomicity, the validity of the covering law) and the question of the existence of the orthocomplementation in the logic, which is here shown to be of secondary importance for the axiomatics.

2. Basic axioms and definitions

We assume that to every physical system there corresponds a pair (L, S) consisting of two sets L and S, whose elements are called events (propositions, questions, yes–no measurements) and states, respectively, and impose the following postulates, which relate L to S.

(A1). With every pair $a \in L$, $m \in S$ there is associated a real number (a, m) belonging to the interval $[0, 1]$, which we interpret as the probability of the occurrence of an event a in a state m.

(A2). If $(a, m) = (b, m)$ for all $m \in S$, then $a = b$.

Owing to the axiom (A2) one can regard any event $a \in L$ as the function from S to $[0, 1]$ defined by $a(m) = (a, m)$, $m \in S$. Define now orthogonality and partial ordering in L by putting

$$\left.\begin{array}{l} a \perp b \text{ iff } a(m) + b(m) \leq 1 \\ a \leq b \text{ iff } a(m) \leq b(m) \end{array}\right\} \quad \text{for all } m \in S.$$

For every $M \subseteq L$ we define $M^\perp$ to be the set of all $a \in L$ such that $a \perp b$ for all $b \in M$. For the case $M = \{a\}$, we will write $a^\perp$ instead of $\{a\}^\perp$. Observe that $a \leq b$ leads immediately to $b^\perp \subseteq a^\perp$.

We will call an event $a \in L$ *empty* (or *zero*), and denote it by 0, if $(a, m) = 0$ for all $m \in S$. An event $b \in L$ is said to be *trivial*, and we denote it by 1, if $(b, m) = 1$ for every $m \in S$. Note that whenever 0 and 1 exist in L, they are obviously unique, and for all $a \in L$ we have

(i) $0 \leq a \leq 1$,

(ii) $0 \perp a$,

(iii) $1 \not\perp a$, provided $a \neq 0$.

Our next axiom is (compare Guz 1978a)

(A3). There is a subset $P \subseteq S$, whose members we will call *pure states*, with the following properties:

(i) $a \not\perp b$ implies $a(p) = 1$ and $b(p) > 0$ for some $p \in P$;

(ii) $a \not\leq b$ implies $a(q) > 0$ and $b(q) = 0$ for some $q \in P$;

(iii) for every $p \in P$ there exists $a \in L$ such that $a(p) = 1$ and $a(q) < 1$ for all $q \in P$, $q \neq p$;

(iv) for every non-zero $a \in L$ there is a $p \in P$ such that $a(p) = 1$.

We shall call the set L the *logic of events* (or the *logic of propositions*) associated with a physical system under study.

Remark. Note that by (A3(iv)) the orthogonality defined above is *proper* i.e. we have $a \not\perp a$ for every non-zero $a \in L$.

Now let m_1 and m_2 be two arbitrary states of the physical system; then we will call the number

$$(m_2 : m_1) = \inf\{a(m_2) \colon a \in L,\ a(m_1) = 1\}$$

the *degree of dependence of m_2 on m_1* (Guz 1975b).

One can easily show (Guz 1975b) that when m_1 and m_2 are the usual quantum mechanical pure states (i.e., the rays in a Hilbert space), the number $(m_1 : m_2)$ gives us the well-known transition probability between m_1 and m_2, and when m_1 and m_2 are mixed ones (that is, the von Neumann density operators), the number $(m_1 : m_2)$ coincides with the so-called semi-inner product between m_1 and m_2, the latter being defined by (Kossakowski 1972)

$$[m_1, m_2] = \|m_2\| \operatorname{Tr}(m_1 \operatorname{sgn} m_2),$$

where $\|\cdot\|$ stands for the trace-norm of the Banach space $T_c(H)$ of the trace-class operators acting in the Hilbert space H (see e.g. Schatten 1960), and sgn m $(m \in T_c(H))$ is defined as

$$\operatorname{sgn} m = \int_{-\infty}^{\infty} \operatorname{sgn} t \, E(\mathrm{d}t),$$

where E is the spectral measure of m.

In the general axiomatic framework based on postulates (A1)–(A3) presented here one immediately finds that, for any two pure states $p, q \in P$, $(p : q) = (s(q))(p)$, where $s(q)$, the so-called support of q, is defined below.

Thus, the transition probability defined here is evidently non-symmetric with respect to the variables p and q.

Now, we shall show some consequences of axioms (A1)–(A3). Let $a \in L$ and $m \in S$. We call the event $a \in L$ a *support* (or *carrier*) of a state $m \in S$ (see Zierler 1961, Pool 1968), if

(i) $a(m) = 1$,

(ii) $a \not\perp b$ implies $b(m) > 0$.

We shall prove that:

(1) Each pure state $p \in P$ has a support $a \in L$, and $a(q) < 1$ for every pure state $q \neq p$.

Proof. Let $p \in P$; by (A3(iii)) there exists $a \in L$ such that $a(p) = 1$ and $a(q) < 1$ for all pure states $q \neq p$. Let $b \in L$, $b \not\perp a$. By (A3(i)) and (A3(iv)) there is an $r \in P$ such that $a(r) = 1$ and $b(r) > 0$, but, because of (A3(iii)), $r = p$. Thus, we have shown that

$$\forall p \in P \quad \exists a \in L \text{ such that } a(p) = 1 \qquad \text{and} \qquad \forall b \in L \text{ such that } b \not\perp a,\ b(p) > 0,$$

that is, the event a is a support of p.

At the same time we proved (see above) that $a(q) < 1$ for every pure state $q \neq p$. The proof of the statement is thus complete.

(2) $a \leq b$ iff $a^\perp \supseteq b^\perp$, and therefore $a^\perp = b^\perp$ leads to $a = b$.

Proof. It needs to be shown that $a^\perp \supseteq b^\perp$ implies $a \leq b$. Suppose that $c \perp b$ always implies $c \perp a$; then, in particular, $s \perp b$ implies $s \perp a$ for all supports s of pure states, which exist by (1), but this means, by the definition of the support, that for every $p \in P$ with $b(p) = 0$ we have also $a(p) = 0$; and hence $a \leq b$ by (A3(ii)), as claimed.

(3) The support of m, whenever it exists, is uniquely determined by the state m. Moreover, it is the smallest element of the set $\{b \in L : b(m) = 1\}$.

Proof. Note that when $a \in L$ is a support of m, then $a^\perp = \{b \in L\colon b(m) = 0\}$; hence the uniqueness of a follows from (2).

Suppose now that $b(m)=1$, $b\in L$, and let $c\perp b$; then, obviously, $c(m)=0$, and hence $c\perp a$ (see the definition of the support). We have thus proved that $b^{\perp}\subseteq a^{\perp}$ and hence $a\leqslant b$ by (2). Our statement is therefore proved.

The support of m, provided it exists, will be denoted by $s(m)$.

(4) $a\nleqslant b\Rightarrow\exists p\in P$ such that $a(p)=1$ and $b(p)<1$ or, equivalently,

$$(\forall p\in P \qquad a(p)=1\Rightarrow b(p)=1)\Rightarrow a\leqslant b.$$

Proof. Suppose $a\nleqslant b$; then $a^{\perp}\nsupseteq b^{\perp}$ by (2), and therefore there exists $c\in L$, $c\perp b$, such that $c\not\perp a$; hence one finds by (A3(i)) a pure state $p\in P$ with $a(p)=1$ and $c(p)>0$; hence $b(p)<1$, as $c\perp b$ implies $b(p)\leqslant 1-c(p)$.

(5) The correspondence $s\colon p\to s(p)$ is one–one.

Proof. Indeed, suppose that $s(p)=s(q)$ for some $p, q\in P$; then $(s(p))(q)=(s(q))(q)=1$, and hence $q=p$ by (1).

Note that owing to (A3(ii)) every event $a\in L$ can be considered as a function on P. By (A3(ii)) and (A3(i)) we have also:

$$a\perp b \text{ iff } a(p)=1 \text{ implies } b(p)=0,$$

$$a\leqslant b \text{ iff } b(p)=0 \text{ implies } a(p)=0,$$

and by (4) we obtain

$$a\leqslant b \text{ iff } a(p)=1 \text{ implies } b(p)=1.$$

We are now in a position to introduce some useful definitions.

Suppose (L, P) to be a pair consisting of a set P together with a set L of functions from P to $[0, 1]$ (that is, $L\subseteq[0, 1]^{P}$), and then define the following:

(a) a is said to be *disjoint* with b($a\llcorner b$ in symbols; $a, b\in L$), if $a(p)=1$ implies $b(p)=0(p\in P)$;

(b) We shall say that a is *orthogonal* to $b(a, b\in L)$, and write $a\perp b$, if $a\llcorner b$ and $b\llcorner a$;

(c) We shall write $a\leqslant_0 b$ and $a\leqslant_1 b$ respectively, if $a^0\supseteq b^0$ and $a^1\subseteq b^1$ respectively, where the following abbreviations are used: x^0 stands for the set $\{p\in P\colon x(p)=0\}$ and x^1 for $\{p\in P\colon x(p)=1\}$, $x\in L$;

(d) We shall say that a *implies* b, or that a is *stronger* than b, and write $a\leqslant b$, if $a\leqslant_0 b$ and $a\leqslant_1 b$.

It can easily be verified that $a\leqslant b\perp c$ leads to $a\perp c$. In other words, we have $b^{\perp}\subseteq a^{\perp}$ whenever $a\leqslant b$.

We shall call the pair (L, P), where $L\subseteq[0, 1]^{P}$, an *event–phase-space structure*, if the following hold for (L, P):

(EPS 0): For every non-zero $a\in L$ there exists $p\in P$ such that $a(p)=1$;

(EPS 1): $a\llcorner b$ implies $b\llcorner a$, and therefore $a\perp b$ iff $a(p)=1$ implies $b(p)=0$;

(EPS 2): $a\leqslant_0 b$ and $b\leqslant_0 a$ leads to $a=b$;

(EPS 3): For every $p\in P$ there exists $a\in L$ such that $a(p)=1$ and $a(q)<1$ for all $q\in P$, $q\neq p$.

We will call the elements of L and P, as before, *events* and *pure states* respectively; the sets L and P will be called the *logic of events* and the *phase space* of the physical system under study.

Note that having assumed (EPS 0, 1, 2, 3) for (L, P) we are in a position to reproduce all the statements (1)–(5), but now with $\leqslant$ replaced by $\leqslant_0$. Thus we have:

(EPS 4) $a \leqslant_0 b$ iff $a^\perp \supseteq b^\perp$;

(EPS 5) $a^1 \subseteq b^1$ implies $a \leqslant_0 b$;

and, obviously, the support of p is now the smallest element of the set $\{b \in L\colon b(p) = 1\}$ in the sense of the partial ordering $\leqslant_0$. Later on, we shall write $\leqslant$ instead of $\leqslant_0$.

Remark. For an event–phase-space structure, we take as the definition of the transition probability

$$(p\colon q) = (s(q))(p),$$

where $s(q)$ is the support of q.

Note, further, that if we assume additionally the postulate

(EPS 3′): $a \leqslant_0 b$ implies $a^1 \subseteq b^1$,

then, by repeating the arguments used by us previously (see Guz 1978a), we can deduce the following facts.

(i) The logic of events $(L, \leqslant)$ is atomic, and $s\colon p \to s(p)$ is a one-to-one mapping of the set P of pure states onto the set $A(L)$ of all atoms of the logic L.

(ii) For every non-zero event $a \in L$ one has

$$a = \bigvee\{s(p)\colon p \in a^1\},$$

and hence one finds L to be atomistic, as every non-zero event $a \in L$ is the least upper bound of the atoms $s(p)$ contained in it.

Remark. (EPS 3′) implies, obviously, the coincidence of the partial orderings $\leqslant$ (which itself is equivalent to $\leqslant_1$ by (EPS 5)) and $\leqslant$.

3. Two extensions of the logic of events

Suppose that (L, P) is an arbitrary event–phase-space structure, and define $\tilde{L}$ as the family of all subsets $M \subseteq L$ such that $M = M^{\perp\perp}$. Obviously, $M \subseteq M^{\perp\perp}$ for every subset $M \subseteq L$, and it is not difficult to check (see e.g. MacLaren 1964) that under the set inclusion L becomes a complete orthocomplemented lattice with joins and meets given by

$$\bigvee_j M_j = \Big(\bigcup_j M_j\Big)^{\perp\perp} \qquad \text{and} \qquad \bigwedge_j M_j = \bigcap_j M_j$$

($\{M_j\}$ is an arbitrary family of members of $\tilde{L}$), and with the orthocomplementation given by $M \to M^\perp (M \in \tilde{L})$.

Moreover, the mapping $i\colon a \to a^{\perp\perp}$, $a \in L$, gives us the embedding of $(L, \leqslant, \perp)$ into $(L, \subseteq, \perp)$, as it is easily seen to have the properties of an orthoinjection, i.e.

(i) $a \leqslant b$ iff $i(a) \subseteq i(b)$,

(ii) $a \perp b$ iff $i(a) \perp i(b)$.

Note also that $\tilde{L}$ does not coincide now with the so-called completion by cuts of L, as was the case when L possessed an orthocomplementation and when the orthogonality was defined by $a \perp b$ iff $a \leqslant b^1$ (see Bugajska and Bugajski 1973c, MacLaren 1964). Only the following can now easily be established (we use here the notation from Bugajska and Bugajski 1973c, i.e. for $M \subseteq L$ we write $M^\nabla = \{a \in L\colon a \geqslant b \text{ for all } b \in M\}$ and $M^\Delta = \{a \in L\colon a \leqslant b \text{ for all } b \in M\}$):

(i) For every $a \in L$ we have $a^{\perp\perp} = a^{\Delta} = a^{\nabla\Delta}$.

(ii) For every $M \subseteq L$ we have $M^{\nabla} = \{a \in L\colon a^{\perp\perp} \supseteq M\}$; hence $M^{\nabla\Delta} = \bigcap_{a \in M^{\nabla}} a^{\perp\perp} \supseteq M^{\perp\perp}$, and therefore $M^{\nabla\Delta} = M$ implies $M^{\nabla\Delta} = M^{\perp\perp} = M$; but, in general, $M^{\nabla\Delta} = M^{\perp\perp}$ does not hold for all $M \subseteq L$.

(iii) If one assumes that for every $a \in L$ there exists a $b \in L$ such that $a^{\perp} = b^{\perp\perp}$, which is easily seen to be equivalent to the existence of an orthocomplementation in L, then $M^{\nabla\Delta} = M^{\perp\perp}$ for an arbitrary $M \subseteq L$, that is, $\tilde{L}$ then coincides with the completion by cuts of L.

Also $(L, \leqslant, \perp)$ may be embedded into an orthocomplemented complete lattice. This extension is realised as follows. Define the orthogonality in the set P of pure states by putting

$$p \perp q \text{ iff } a(p) = 1,\ a(q) = 0 \text{ and } b(p) = 0,\ b(q) = 1$$

for some $a, b \in L$, and, for any $Q \subseteq P$, define $Q^{\perp}$ to be the set of all $p \in P$ such that $p \perp q$ for all $q \in Q$. Obviously, $Q \subseteq Q^{\perp\perp}$ for every $Q \subseteq P$. Let $C(P, \perp) = \{Q \subseteq P\colon Q = Q^{\perp\perp}\}$. The family $C(P, \perp)$, called the *phase geometry* associated with the physical system under study (see Guz 1975a, 1978a), becomes a complete orthocomplemented lattice with joins and meets given by

$$\bigvee_j Q_j = \Big(\bigcup_j Q_j\Big)^{\perp\perp} \qquad \text{and} \qquad \bigwedge_j Q_j = \bigcap_j Q_j$$

($\{Q_j\}$ is any family of members of $C(P, \perp)$), and with the orthocomplementation defined by $Q \to Q^{\perp}$, $Q \in C(P, \perp)$.

Remark. For the empty set $\varnothing$ we put, by definition, $\varnothing^{\perp} = P$, which leads immediately to $\varnothing, P \in C(P, \perp)$.

Define now the mapping from L to $C(P, \perp)$ by

$$j(a) = a^{1\perp\perp}, \qquad a \in L.$$

One can prove the following properties of the mapping j:

(i) $a \leqslant b$ implies $j(a) \subseteq j(b)$,

(ii) $a \perp b$ implies $j(a) \perp j(b)$,

(iii) $j(0) = \varnothing$ and $j(1) = P$, provided 0 and 1 exist in L.

The properties (i) and (iii) are obvious. In order to show (ii), it is sufficient to note that $a \perp b$ leads to $a^1 \perp b^1$, and the latter we prove as follows: $p \in a^1$, $q \in b^1$ imply $s(p) \leqslant a$ and $s(q) \leqslant b$; hence $s(p) \perp s(q)$, as $a \perp b$, and hence $p \perp q$ (see the lemma below), which shows that $a^1 \perp b^1$, as claimed.

Lemma 3.1. For $p, q \in P$ we have $p \perp q$ if and only if $s(p) \perp s(q)$.

Proof. Assume first that $p \perp q$; then, by the definition, there are $a, b \in L$ such that $a(p) = 1$, $a(q) = 0$ and $b(p) = 0$, $b(q) = 1$; hence, for example, $s(p) \leqslant a$ and $s(q) \perp a$, which leads to $s(p) \perp s(q)$. The converse statement is obvious, as we have, when $s(p) \perp s(q)$,

$$(s(p))(p) = 1,\ (s(q))(p) = 0,\ (s(q))(q) = 1, \text{ and } (s(p))(q) = 0,$$

which means that $p \perp q$ as desired.

Since, in general, j is now not an orthoinjection, we find $C(P, \perp)$ to be inappropriate as the extension of the logic $(L, \leqslant, \perp)$, although for the case where L is orthocomplemented, it can be shown (Guz 1978b) that $C(P, \perp)$ is orthoisomorphic to $\tilde{L}$, and therefore $C(P,\perp)$ and $\tilde{L}$ are then equally good as extensions of L.

4. Further axioms and their consequences

Suppose (L, P) to be an arbitrary event–phase-space structure, and assume, after Mackey (see e.g. Mackey 1963), the so-called 'orthogonality postulate':

(B1) For every (finite or denumerable) sequence $\{a_i\}$ of pairwise orthogonal events, its sum $\Sigma_i\, a_i$ belongs to L.

Remark. When $\{a_i\}$ is infinite, the sum $\Sigma_i\, a_i$ is meant in the usual sense of pointwise convergence of the series, i.e.

$$\left(\sum_i a_i\right)(p) = \sum_i a_i(p)$$

for all $p \in P$.

The sum $\Sigma_i\, a_i$ is easily shown to be the least upper bound of all a_i's (in the sense of the partial ordering $\leqslant$). This is the content of the following statement.

Proposition 4.1. $\Sigma_i\, a_i = \bigvee_i a_i$, where a_i are mutually orthogonal events.

Before proving this proposition, we need to prove a lemma.

Lemma 4.2. Let $a = \Sigma_i\, a_i$ for some orthogonal sequence $\{a_i\} \subseteq L$. Then $b \perp a$ if and only if $b \perp a_i$ for each $i = 1, 2, \ldots$, or, in other words, $a^\perp = \bigcap_i a_i^\perp$.

Proof. The 'only if' part of the lemma is obvious, since by (B1) we have $a \geqslant a_i$ for all i, and hence (see (EPS 4)) $a^\perp \subseteq a_i^\perp$ for all i.

To prove the 'if' part, assume $b \perp a_i$ for every i, and let $a = \Sigma_i\, a_i$. By applying the axiom (B1) to the orthogonal sequence $\{b, a_1, a_2, \ldots\}$ we find

$$(b + a_1 + a_2 + \ldots)(p) = b(p) + \sum_i a_i(p) = b(p) + a(p)$$

for all $p \in P$, and hence $b(p) + a(p) \leqslant 1$ for all $p \in P$; hence $b \perp a$.

The lemma is thus proved.

The proof of proposition 4.1 is now straightforward. As we know that $a \geqslant a_i$ for all i, it remains to be shown that $b \geqslant a$ for every b satisfying $b \geqslant a_i$ for all $i = 1, 2, \ldots$ But $b \geqslant a_i$ (all i) leads to $b^\perp \subseteq \bigcap_i a_i^\perp = a^\perp$, where the last equality holds by lemma 4.2; and hence $b \geqslant a$ by (EPS 4).

This completes the proof of proposition 4.1.

Now we assume the postulate which is known as the *orthomodularity* of the logic of events:

(B2) If $a \leqslant b (a, b \in L)$, then $b - a \in L$.

Remark. By (B1) we have $a \perp b$ iff $a(p)+b(p) \leqslant 1$ for every $p \in P$, and by (B2) we obtain $a \leqslant b$ iff $a(p) \leqslant b(p)$ for all p. Hence, in particular, $\leqslant$ and $\leqslant$ coincide, and as a consequence the statements (i) and (ii) given at the end of § 2 hold for (L, P). In the sequel we will always, in this section, write $\leqslant$ to denote the partial ordering in L.

Our next axiom is the so-called 'projection postulate' (compare Bugajska and Bugajski 1973b, c), an abstract form of the famous von Neumann projection postulate (von Neumann 1932).

(B3) If for $a \in L$ and $p \in P$ we have $0 < a(p) < 1$, then there exist atomic events $e \leqslant a$ and $f \perp a$ such that $e(p) = a(p)$ and $f(p) = 1 - a(p)$.

We complete our list of axioms by assuming the following:

(B4) If $0 < (p\colon q) < 1$ for some $p, q \in P$, then there is the unique pure state $r \in P$ such that $r \perp q$ and $(p\colon q)+(p\colon r) = 1$. Moreover, if additionally $a(p) = a(q) = 0$ for some $a \in L$, then also $a(r) = 0$.

Lemma 4.3. Assume the validity of axioms (B1)–(B4) for an event–phase-space structure (L, P). Then, for any pair of atoms e, $f \in A(L)$ there exists $e \vee f$ in L. Moreover, one then has $e \vee f = g + f$ for some atom g, $g \perp f$, provided $e \neq f$.

Proof. Let $e, f \in A(L)$, and let $p = s^{-1}(e)$, $q = s^{-1}(f)$ (see statement (i) at the end of § 2). One can assume without any loss of generality that $0 < (p\colon q) < 1$, since $(p\colon q) = 0$ implies $s(p) \perp s(q)$ or $e \perp f$, and the existence of $e \vee f (= e + f)$ then follows from axiom (B1) and proposition 4.1; similarly, $(p\colon q) = 1$ leads to $s(p) \leqslant s(q)$, and we then have $e \vee f = e = f$.

By (B4) there then exists the unique pure state $r \perp q$ such that $(p\colon q)+(p\colon r) = 1$, and the latter equality may be written as $(f + s(r))(p) = 1$, since $f = s(q) \perp s(r)$ (see lemma 3.1), and therefore $e = s(p) \leqslant f + s(r)$.

We shall show that $f + s(r) = e \vee f$. Suppose that $a \geqslant e, f$; one then needs to prove that $a \geqslant f + s(r)$. By the orthomodularity of L one can write $a = f + c$, where c (being actually the difference $a - f$) is orthogonal to f, and $a \geqslant e = s(p)$ leads then to $f(p) + c(p) = a(p) = 1$; hence $c(p) = 1 - f(p) \neq 0$, 1, and therefore, by the first half of axiom (B3) one can choose an atom $g \leqslant c$ such that $g(p) = c(p) = 1 - f(p)$. For the pure state $r' = s^{-1}(g)$ we then have $(p\colon r') + (p\colon q) = 1$. On the other hand, $g \leqslant c$ implies $g \perp f$, since $c \perp f$, and therefore by the uniqueness requirement in (B4) we obtain $r' = r$, and hence $g = s(r') = s(r)$.

Finally $a \geqslant c \geqslant g = s(r)$ and $a \geqslant f$ imply $a \geqslant s(r) + f$, as desired. The proof of the lemma is thus complete.

Corollary 4.4. If e and f are atoms in L such that $e, f \perp a$ for some $a \in L$, then $e \vee f \perp a$ also.

Proof. One can assume, obviously, that $e \neq f$. Then, by lemma 4.3 we have $e \vee f = g + f$ for $g = s(r)$, where r is chosen, for given $p = s^{-1}(e)$ and $q = s^{-1}(f)$, according to the prescription of axiom (B4). But e, $f \perp a$ means that $a(p) = a(q) = 0$; hence by the second half of (B4) one finds $a(r) = 0$, and hence $g = s(r) \perp a$. Finally, $g \perp f$, $g \perp a$ and $f \perp a$ imply $g + f = e \vee f \perp a$, by lemma 4.2.

Lemma 4.5. Assume the validity of axioms (B1)–(B4) for an event–phase-space structure (L, P). Then the atoms e and f in (B3) are determined uniquely by a and p.

Proof. Let us suppose that for two atoms $e_1, e_2 \leqslant a$ we have $e_1(p) = e_2(p) = a(p)$, where $a(p) \neq 0$, 1; then, obviously, $e_1(p) = e_2(p) = (e_1 \vee e_2)(p) = a(p) \neq 0$, since $e_1 \vee e_2 \leqslant a$ ($e_1 \vee e_2$ exists by lemma 4.3). Similarly, if for two atoms e_1, $e_2 \perp a$ one has $e_1(p) = e_2(p) = 1 - a(p)$, one finds also $e_1(p) = e_2(p) = (e_1 \vee e_2)(p) \neq 0$, as then $e_1 \vee e_2 \perp a$ by corollary 4.4, which implies $(e_1 \vee e_2)(p) + a(p) \leqslant 1$; hence $1 - a(p) = e_1(p) = e_2(p) \leqslant (e_1 \vee e_2)(p) \leqslant 1 - a(p)$, and hence the desired equality follows.

Thus it remains to be shown that any two atoms e_1, e_2 satisfying $e_1(p) = e_2(p) = (e_1 \vee e_2)(p) \neq 0$ are equal. Suppose the contrary, that is, there exist two distinct atoms $e_1 \neq e_2$ such that the above equalities hold. By lemma 4.3 one can write $e_1 \vee e_2 = e_1 + f_1 = e_2 + f_2$, where f_i are atoms such that $f_i \perp e_i$ $(i = 1, 2)$. Then, in particular, $e_1 \vee e_2 \geqslant f_1 \vee f_2$, and hence by applying the orthomodularity of L one finds $e_i + f_i = e_1 \vee e_2 = f_1 \vee f_2 + c (c \in L, c \perp f_1 \vee f_2)$, hence $e_i = (f_1 \vee f_2 - f_i) + c$, which leads to $c \leqslant e_i (i = 1, 2)$, and hence $c = 0$, since we assumed that $e_1 \neq e_2$. Thus $e_1 \vee e_2 = f_1 \vee f_2$. Note that $f_1 \neq f_2$, as $f_1 = f_2$ would imply $e_1 \vee e_2 = f_i$; hence $e_1 = e_2 = f_i$, which contradicts our assumption. On the other hand, $e_1 \vee e_2 = e_i + f_i (i = 1, 2)$, which implies $e_1(p) = e_2(p) = (e_1 \vee e_2)(p) = e_i(p) + f_i(p)$, $i = 1$, 2; hence $f_i(p) = 0$ for each $i = 1$, 2. Hence $s(p) \perp f_i (i = 1, 2)$, which leads to $s(p) \perp f_1 \vee f_2$ by corollary 4.4, hence $(f_1 \vee f_2)(p) = 0$, and therefore $(e_1 \vee e_2)(p) = 0$—a contradiction. The proof of the lemma is therefore complete.

Thus, as a consequence of axioms (B1)–(B4) we found the following:

(B3′) $0 < a(p) < 1$ implies $a(p) = e(p) = 1 - f(p)$ for two unique atomic events $e \leqslant a$ and $f \perp a$, or, equivalently,

(B3″) $0 < a(p) < 1$ implies $a(p) = (p\colon q) = 1 - (p\colon r)$ for two unique pure states q and r satisfying $a(q) = 1$ and $a(r) = 0$.

We shall refer to (B3′) or (B3″) as the standard form of the projection postulate, and we will denote q (see (B3″)) by p_a and r by p^a. Thus, we have $a(p_a) = 1$, $a(p^a) = 0$ (or, equivalently, $s(p_a) \leqslant a$ and $s(p^a) \perp a$) and $a(p) = (p\colon p_a) = 1 - (p\colon p^a)$ for all p with $a(p) \neq 0$, 1. Note also that $p_a \perp q^a$ for all $p, q \in P$ for which $p(a), q(a) \neq 0$, 1.

Lemma 4.6. Assume the validity of axioms (B1), (B2) and (B3′) for an event–phase-space structure (L, P). Then, for each $e \in A(L)$ and $a \in L$ there exists $e \vee a$ in L. Moreover, if $e \nleqslant a$, then $e \vee a = f + a$ for some $f \in A(L)$, $f \perp a$.

Proof. Let $a \in L$, $e \in A(L)$, and let $p = s^{-1}(e)$. One can assume, without any loss of generality, that $0 < a(p) < 1$, since $a(p) = 0$ implies $e = s(p) \perp a$; the existence of $e \vee a$ then follows from (B1) and proposition 4.1, and $a(p) = 1$ leads to $e = s(p) \leqslant a$ and then $e \vee a = a$.

Then, by the second half of (B3′), there exists a unique $f \in A(L)$ such that $f \perp a$ and $f(p) = 1 - a(p)$; hence $(f + a)(p) = 1$ and hence $e = s(p) \leqslant f + a$.

We shall show that $f + a = e \vee a$. Let $b \geqslant e$, a; one needs to prove that $b \geqslant f + a$. By the orthomodularity of L one can write $b = a + c$, where $c \perp a$. Further, $b \geqslant e = s(p)$ leads to $a(p) + c(p) = b(p) = 1$ and hence $c(p) = 1 - a(p) \neq 0$, 1. By the first half of axiom (B3′) one can then choose $g \in A(L)$ with $g \leqslant c$ and $g(p) = c(p) = 1 - a(p)$. But $g \leqslant c$ implies $g \perp a$, since $c \perp a$, and therefore by the uniqueness part of (B3′) one finds $g = f$. Finally, $b \geqslant c \geqslant g = f$ and $b \geqslant a$ imply $b \geqslant f + a$, as desired. This completes the proof of the lemma.

If for all $a \in L$ and $e \in A(L)$ there exists $a \vee e$ in L, we will then say that the *atom adjoining* holds in L.

Theorem 4.7. Assume for an event–phase-space structure (L, P) the validity of axioms (B1), (B2), (B4) together with the first half of the projection postulate (B3), i.e. suppose that $0<a(p)<1(a\in L, p\in P)$ implies the existence of an atom $e\leqslant a$ with $e(p)=a(p)$. Then the second half of this postulate is equivalent to any of the following conditions.

(1) In L, both the atom adjoining and the following property hold: let G be a finite set of atomic events, say, $G=\{e_1,\ldots,e_n\}$ and let e be an atom such that $e\leqslant\bigvee G$ and $e\not\leqslant\bigvee(G\backslash\{e_j\})$ for every $j=1,\ldots,n$; then for any partition $I\cup J=\{1,2,\ldots,n\}$, $I\cap J=\varnothing$, of the index set $\{1,2,\ldots,n\}$ there exists an atom $f\in L$ such that

$$f\leqslant e\vee\bigvee_{i\in I}e_i \qquad \text{and} \qquad f\leqslant\bigvee_{j\in J}e_j.$$

(2) In L, besides the atom adjoining, the following holds: for any four atoms e, e_1, e_2, $e_3\in A(L)$ (not necessarily all distinct) such that $e\leqslant e_1\vee e_2\vee e_3$, $e\neq e_3$, and $e\not\leqslant e_1\vee e_2$, there exists an atom $f\in A(L)$ such that $f\leqslant e\vee e_3$ and $f\leqslant e_1\vee e_2$.

(3) In L, both the atom adjoining and the following property are fulfilled: for any $a\in L$ and any atom $e\in L$ not contained in a, $a\vee e-a$ is also an atom.

(4) Besides the atom adjoining, the following is fulfilled in L: for any $a\in L$, $a\neq 0, 1$, and any atom $e\in L$ there exist two atoms e_1, $e_2\in L$ such that $e_1\leqslant a$, $e_2\perp a$ and $e\leqslant e_1\vee e_2$.

(5) In L, besides the atom adjoining, the covering law holds, that is, for any $a\in L$ and any atom $e\in L$, $a\vee e\geqslant b\geqslant a$ implies either $b=a$ or $b=a\vee e$.

(6) In L, the atom adjoining holds, and there exists a dimension function on L_f, the set of finite elements† of L, that is, a function d from L_f to the positive integers $\{1,2,\ldots\}$ possessing the properties

(i) d is strictly increasing, i.e., $d(a)<d(b)$, whenever $a<b(a, b\in L_f)$,

(ii) $d(a\vee b)+d(a\wedge b)=d(a)+d(b)$ for each pair a, $b\in L_f$ for which $a\wedge b$ exists‡.

(7) Besides the atom adjoining, the following holds for L: for any three finite events a, b, $c\in L$ such that $a\leqslant c$ and $b\wedge c=0$, one has $a=(a\vee b)\wedge c$, provided $(a\vee b)\wedge c$ exists in L.

Proof. We will prove the theorem as the following chain of implications: (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$ the second half of (B3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(7)$\Rightarrow$(1).

As the proof of the implications (1)$\Rightarrow$(2)$\Rightarrow$(3) and (4)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(7)$\Rightarrow$(1) is simply a repetition (after some minor modifications) of that given by us previously (see Guz 1978b), we need only to prove the implications (3)$\Rightarrow$ the second half of (B3)$\Rightarrow$(4).

Assume the validity of (3) and then prove the second part of the projection postulate (B3). Suppose that $0<a(p)<1$ for some $a\in L$ and $p\in P$, and let $e=s(p)$. Since $e\not\leqslant a$, we obtain $e\vee a-a\in A(L)$ by (3). We also have $(e\vee a-a)(p)=(e\vee a)(p)-a(p)=1-a(p)$, which is just the second half of (B3).

Assume now the second part of (B3); both parts of the projection postulate are now valid, and we prove (4). Let $a\in L$, $a\neq 0$, 1, and let $e\in A(L)$. Obviously, one can assume that $e\not\leqslant a$ and $e\not\perp a$. For $p=s^{-1}(e)$ one then easily obtains $0<a(p)<1$, and therefore there exist by (B3) two atoms $e_1\leqslant a$ and $e_2\perp a$ such that $e_1(p)=a(p)$ and $e_2(p)=1-a(p)$; hence, by using the fact that $e_1\perp e_2$, we find $(e_1\vee e_2)(p)=e_1(p)+e_2(p)=1$, which leads to $e=s(p)\leqslant e_1\vee e_2$ as desired.

The theorem is therefore proved.

† An event $a\in L$ is said to be finite, if it is a join of a finite number of atoms.

‡ $a\wedge b$ is then finite, provided $a\wedge b\neq 0$. For the zero event we put, by definition, $d(0)=0$. Note also that $a\vee b$ always exists for finite a, $b\in L$, owing to the atom adjoining in L.

5. Filters on the transition probability space

It is convenient to adjoin to $P((L, P)$ being an arbitrary event–phase-space structure) some fictitious 'pure' state, called the *zero state* and denoted by 0, which is defined by the requirement that $a(0) = 0$ for all $a \in L$. More precisely, in order to define the zero state 0 one can use the observation that every pure state $p \in P$ may be regarded as a function on L, owing to the identification of p with the function $\hat{p}$: $a \to a(p)$. Indeed, to see that the correspondence $p \to \hat{p}$ is one–one, suppose that $p = q$ for some $p, q \in P$; then, in particular, $(s(q))(p) = (s(q))(q) = 1$ and hence $p = q$, as $(s(q))(p) < 1$ for all $p \neq q$. Now, we define the zero state 0 as the zero function on L. Moreover, after identifying P with the set $\hat{P} = \{\hat{p}: p \in P\}$ and considering the events from L as the functions on $\hat{P} \cup \{0\}$ (by putting $a(\hat{p}) = a(p)$ and $a(0) = 0$), we also extend L by adjoining to it some new event 0, which we define as the zero function on $\hat{P} \cup \{0\}$, and therefore 0 is the smallest element of the extended logic $L \cup \{0\}$.

The event 0 we will call the *zero* or *impossible* event.

For the sake of brevity, we shall write P_0 instead of $\hat{P} \cup \{0\}$ in the sequel and, similarly, L_0 instead of $L \cup \{0\}$.

It is also convenient to extend the transition probability function onto a whole P_0 by putting $(0: p) = (p: 0) = 0$ for all $p \in P_0$. Note that for the extended transition probability the previous formula will hold, i.e., $(p: q) = (s(q))(p)$, if we put, by definition, $s(0) = 0$. For the zero state we also put, by definition, $0 \perp p$ for all $p \in P_0$.

One can easily show the following properties of the transition probability:

(i) $0 \leqslant (p: q) \leqslant 1$ for all $p, q \in P_0$,

(ii) $(p: q) = 0$ iff $p \perp q$,

(iii) $(p: q) = 1$ iff $p = q$.

In the remainder of this section we will assume axioms (EPS 3′) and (B3′) for (L, P). Any event–phase-space structure satisfying the above axioms will be called the *standard* one.

Having assumed (B3′) for (L, P), one can define two families of mappings E_a, F_a: $P_0 \to P_0$, both indexed by members of L_0, by putting

$$E_a p = \begin{cases} p_a & \text{if } a(p) \neq 0, 1 \\ p & \text{if } a(p) = 1 \\ 0 & \text{if } a(p) = 0 \end{cases}$$

and

$$F_a p = \begin{cases} p^a & \text{if } a(p) \neq 0, 1 \\ 0 & \text{if } a(p) = 1 \\ p & \text{if } a(p) = 0, \end{cases}$$

and write p_a in place of $E_a p$ and p^a instead of $F_a p$.

Note that now $p_a \perp q^a$ for *all* $p, q \in P_0$, and obviously, $0_a = 0^a = 0$ for every $a \in L_0$. From the definition of E_a and F_a it follows that one can now write

$$a(p) = (p: p_a) = 1 - (p: p^a)$$

for all $p \in P_0$, with only one exception, when we have $p = 0$ in the second equality above.

Note that $E_0 = 0$, where 0 denotes the zero mapping from P_0 to P_0 (defined by $0: p \to 0$ for all $p \in P_0$), and $F_0 = I$, where I is the identity map from P_0 to P_0. Note also that, by definition, $p_a = 0$ iff $a(p) = 0$ iff $(p: p_a) = 0$, and $p^a = 0$ iff either $a(p) = 1$ or $p = 0$ iff $(p: p^a) = 0$.

We shall interpret the transformations E_a and F_a as the *filtering procedures* (briefly, *filters*) corresponding to the event (proposition, yes–no measurement) $a \in L_0$. More precisely, we shall call E_a the *filter*, and F_a the *dual filter* associated with a. Note that the correspondences $a \to E_a$, F_a are one–one. In fact, suppose that $E_a = E_b$ for some a, $b \in L_0$; then we have $(p: p_a) = (p: p_b)$ for all p, or $a(p) = b(p)$ for all p, that is, $a = b$. Similarly, we prove that $F_a = F_b$ implies also $a = b$.

Remark. Note that when we assume all the axioms (B1)–(B4) for an event-phase-space structure (L, P), then $s(p^a) = s(p) \vee a - a$ (where $s(p) \vee a$ exists by lemma 4.6). In fact, if $a(p) \neq 0, 1$, this readily follows from (B3″), as for q with $s(q) = s(p) \vee a - a$ (the latter being an atom by theorem 4.7, (3)) we have $(p: q) = (s(p) \vee a)(p) - a(p) = 1 - a(p)$, which means, by (B3″), that $q = p^a$. If $a(p) = 0$, we have $s(p) \perp a$, and hence $s(p) \vee a - a = s(p) = s(p^a)$, as $a(p) = 0$ implies $p = p^a$; if $a(p) = 1$, we have $s(p) \leqslant a$, which leads to $s(p) \vee a - a = 0 = s(p^a)$ since $a(p) = 1$ implies $p^a = 0$.

Theorem 5.1. For every standard event–phase-space structure (L, P) the following statements are true:

(1) $a(p_a) = 1$, provided $p_a \neq 0$, and $a(p^a) = 0$;

(2) $(p: p_a) = a(p)$ and $(p: p^a) = 1 - a(p)$, the latter being valid only for non-zero p;

(3) $(p: p_a) = 0$ implies $p_a = 0$ and $(p: q_a) = 0$ for all $q \in P_0$, and, similarly, $(p: p^a) = 0$ leads to $p^a = 0$ and $(p: q^a) = 0$ for every $q \in P_0$;

(4) all E_a and F_a are idempotent;

(5) $E_a F_a = F_a E_a = 0$, that is, $(p^a)_a = (p_a)^a = 0$ for all $p \in P_0$;

(6) $(p: p_a) = (p: q_a) \neq 0$ implies $p_a = q_a$, and $(p: p^a) = (p: q^a) \neq 0$ leads to $p^a = q^a$;

(7) $p_{s(q)} = \begin{cases} q, & \text{if } p \not\perp q \\ 0, & \text{if } p \perp q. \end{cases}$

Proof. The statements (1), (2) and the implications $(p: p_a) = 0 \Rightarrow p_a = 0$, $(p: p^a) = 0 \Rightarrow p^a = 0$ are obvious, as they follow directly from the definition of E_a and F_a (see also (B3″)). To prove the remaining part of (3), let us first asume that $(p: p_a) = 0$. Then $a(p) = 0$ by (2), and hence $p^a = p$ (see the definition of F_a), which leads to $(p: q_a) = (p^a: q_a) = 0$ since $p^a \perp q_a$. Suppose now that $(p: p^a) = 0$; then we prove that $(p: q^a) = 0$ for all $q \in P_0$. Of course, one can assume without loss of generality that $p \neq 0$ and $a \neq 0$, as otherwise the statement holds trivially. Then by using (2) one obtains $a(p) = 1$; hence $p_a = p$ (see the definition of E_a), and hence $(p: q^a) = (p_a: q^a) = 0$ as before. Statement (3) is therefore proved.

We shall now prove (4) i.e. the idempotency of E_a and F_a. By (2) and (1) we obtain $(E_a p: E_a^2 p) = (p_a: (p_a)_a) = a(p_a) = 1$, provided $p_a \neq 0$, hence $E_a^2 p = E_a p$. When $p_a = 0$, we have $E_a^2 p = 0 = E_a p$; thus $E_a^2 = E_a$ indeed. Similarly, by (2) and (1) again, we find $(F_a p: F_a^2 p) = (p^a: (p^a)^a) = 1 - a(p^a) = 1$, provided $p^a \neq 0$, and hence $F_a^2 p = F_a p$; for $p^a = 0$ one has $F_a^2 p = 0 = F_a p$, and thus we have shown $F_a^2 = F_a$, as desired.

To prove (5), note that $(p^a: (p^a)_a) = a(p^a) = 0$ by (2) and (1) respectively, and hence we find $(p^a)_a = 0$ by (3). Similarly, by using (2) and (1) we obtain $(p_a: (p_a)^a) = 1 - a(p_a) = 0$ whenever $p_a \neq 0$, and hence $(p_a)^a = 0$ by (3); for $p_a = 0$ we have also $(p_a)^a = 0$. Statement (5) is thus proved.

Note next that $(p: p_a) = (p: q_a) \neq 0, 1$ and $(p: p^a) = (p: q^a) \neq 0, 1$ imply $q_a = p_a$ and $q^a = p^a$ respectively, by the uniqueness requirement in (B3″), since $a(q_a) = 1$ and $a(q^a) = 0$ by (1) (obviously, we also use (2) in the proof). Of course, $(p: p_a) = (p: q_a) =$

$1=(p\colon p^a)=(p\colon q^a)$ implies $p_a=p=q_a$ and $p^a=p=q^a$, and therefore statement (6) is proved.

Finally, let us consider the filter $E_{s(q)}$, where q is a fixed element of P_0, and let $p\in P_0$. By (2) we have $(p\colon p_{s(q)})=(s(q))(p)=(p\colon q)$, and therefore $p\perp q$ iff $(p\colon p_{s(q)})=0$ iff $p_{s(q)}=0$, the latter equivalence holding by (3). Therefore, if $p\not\perp q$, we have $p_{s(q)}\neq 0$, and then (see the definition of filter) $s(p_{s(q)})\leqslant s(q)$; hence $s(p_{s(q)})=s(q)$, as the supports of non-zero pure states are atomic events. Hence $p_{s(q)}=q$, since the mapping s is one–one.

This completes the proof of (7), and, at the same time, the theorem is proved.

We shall now state some facts about the partial ordering and orthogonality in L_0.

Proposition 5.2. For a, $b\in L_0$ the following conditions are mutually equivalent:

(i) $a\leqslant b$,
(ii) $E_bE_a=E_a$,
(iii) $F_aF_b=F_b$,
(iv) $R_a\subseteq R_b$,
(v) $R^a\supseteq R^b$,

where R_x and R^x stand for the ranges of E_x and F_x, respectively.

Proof. The proof of the proposition will consist of the following two chains of implications: (i)$\Rightarrow$(ii)$\Rightarrow$(iv)$\Rightarrow$(i) and (i)$\Rightarrow$(iii)$\Rightarrow$(v)$\Rightarrow$(i).

Let us assume (i) and then prove (ii), i.e. we shall show that $(p_a)_b=p_a$ for all $p\in P_0$. One can assume, without any loss of generality, that $p_a\neq 0$; then $a(p_a)=1$ by theorem 5.1(1), which implies (see statement (2) of theorem 5.1) $(p_a\colon (p_a)_b)=b(p_a)=1$ by (i), and hence $p_a=(p_a)_b$ as desired.

Similarly, having assumed (i) one easily shows that $(p^b)^a=p^b$ (all p); for $p^b=0$ this is trivial, and for $p^b\neq 0$ one has $b(p^b)=0$ (see theorem 5.1(1)) which implies $a(p^b)=0$ by (i); but by statement (2) of theorem 5.1 we have $a(p^b)=1-(p^b\colon (p^b)^a)$, and therefore one obtains $p^b=(p^b)^a$, which proves (iii).

Assume now (ii), and let $p\in R_a$. One can then easily show that $p=p_a=(p_a)_b$, where the last equality holds by (ii), and hence $p\in R_b$, which proves the implication (ii)$\Rightarrow$(iv). The proof of the implication (iii)$\Rightarrow$(v) is essentially the same, since having assumed (iii) we find, for an arbitrary $p\in R^b$, that $p=p^b=(p^b)^a\in R^a$.

Finally, to prove the implications (iv)$\Rightarrow$(i) and (v)$\Rightarrow$(i) it is sufficient to note that for an arbitrary $x\in L_0$ we have

$$R_x=\{p\in P_0\colon x(p)=1\}\cup\{0\}$$

and

$$R^x=\begin{cases}\{p\in P_0\colon x(p)=0\}, \text{ when } x\neq 0\\ \{0\}, \text{ when } x=0.\end{cases}$$

The proof of the proposition is thus complete.

Proposition 5.3. For every $a\in L_0$ we have

$$R_a^{\perp}=\{p\in P_0\colon p_a=0\}=R^a$$

and

$$R^{a\perp}=\{p\in P_0\colon p^a=0\}=R_a.$$

As a corollary one obtains $R_a^{\perp\perp}=R_a$ and $R^{a\perp\perp}=R^a$ for all $a\in L_0$.

Proof. One can assume, without loss of generality, that $a \neq 0$. Let us suppose that $p \in R_a^{\perp}$; then, in particular, $(p: p_a) = 0$, and hence $p_a = 0$ by theorem 5.1(3), which proves the inclusion $R_a^{\perp} \subseteq \{p \in P_0: p_a = 0\}$.

Note further that since $p_a = 0$, $p \neq 0$ imply $(p: p^a) = 1$ (see theorem 5.1(2)) we have $p = p^a$, which shows that $\{p \in P_0: p_a = 0\} \subseteq R^a$, as obviously $0 \in R^a$.

To prove the converse inclusion, observe that the assumption $p \in R^a$ (which means that $p = q^a$ for some $q \in P_0$) implies that $p \perp r$ for all $r \in R_a$, that is, $R^a \subseteq R_a^{\perp}$.

Therefore, we have shown that $R_a^{\perp} = \{p \in P_0: p_a = 0\} = R^a$. The proof of the remaining part of the proposition is very similar, and will therefore be omitted.

Proposition 5.4. For $a, b \in L_0$ the following conditions are equivalent:

(i) $a \perp b$,

(ii) $F_a E_b = E_b$,

(iii) $E_a E_b = 0$,

(iv) $R_a \perp R_b$,

(v) $R_a \subseteq R^b$,

(vi) $R_b \subseteq R^a$.

Proof. Let us assume (i) and then prove (ii). Let p be an arbitrary pure state from P_0; since $a \perp b$ implies $a \perp s(p_b)$, as $s(p_b) \leqslant b$, we have $a(p_b) = 0$, and therefore (see the definition of the dual filter) we obtain $(p_b)^a = p_b$ (all p), which means that $F_a E_b = E_b$, as claimed.

To prove the next implication, (ii)$\Rightarrow$(iii), it is sufficient to appeal to theorem 5.1(5); we then find, by using (ii), that $E_a E_b = E_a F_a E_b = 0$.

Further, having assumed (iii) one easily shows that $R_a \perp R_b$, since for any pair p, $q \in P_0$ one then has $s(p_b) \perp s(q_a)$, as $(p_b)_a = 0$ (all p) leads to $a(p_b) = (p_b: (p_b)_a) = 0$ (see theorem 5.1(2)), and hence $s(p_b) \perp a$ (all p); but at the same time $s(q_a) \leqslant a$ for every $q \in P_0$, and therefore $s(p_b) \perp s(q_a)$ for all p, q indeed. From $s(q_a) \perp s(p_b)$ one obtains $q_a \perp p_b$ (all q, p) by lemma 3.1, that is, $R_a \perp R_b$ as desired.

Furthermore, from proposition 5.3 there readily follows the equivalence of (iv), (v) and (vi). Therefore, in order to close the proof of the proposition, we need to show that (v) (being equivalent to (vi)) implies (i). But this is obvious as $R_a \subseteq R^b$ means that for every $p \in P$ with $a(p) = 1$ we have $b(p) = 0$, that is, $a \perp b$. This completes the proof of the proposition.

We now introduce two definitions:

A pair $(P_0, (:))$ consisting of a set P_0 together with a function $(:): P_0 \times P_0 \to R^1$ is said to be the *transition probability space* (briefly, TPS—compare Guz 1979), if the following conditions are satisfied:

(TP 1) $0 \leqslant (p: q) \leqslant 1$ for all $p, q \in P_0$;

(TP 2) There exists $p_0 \in P_0$ such that $(p_0: p) = (p: p_0) = 0$ for all $p \in P_0$;

(TP 3) For every $p \neq p_0$ we have $(p:p) = 1$;

(TP 4) $(p: q) = 1$ implies $p = q$.

The members of the set P_0 will be called *pure states*, and we shall call the function $(:)$ the *transition probability* in P_0.

The state p_0 in (TP 2) is easily shown to be unique, as, by (TP 3), p_0 is the unique state with $(p_0: p_0) = 0$. We denote it by 0 and call it the *zero state*. Therefore the set P_0 is of the form $P_0 = P \cup \{0\}$, where P stands for the set of non-zero pure states.

The axioms (TP 2), (TP 3) and (TP 4) imply the following property of the transition probability (Guz 1979):

(TP 5) If $(p\colon q)=(p\colon r)$ for all $p\in P_0$, then $q=r$, and similarly, if $(q\colon p)=(r\colon p)$ for every $p\in P_0$, then $q=r$.

With the help of the transition probability we define the *orthogonality* in the set P_0 (Guz 1979): $p\perp q$ iff $(p\colon q)=(q\colon p)=0$.

A set F of mappings from P_0 to P_0 will be called the *logic of filters* (compare Guz 1979), if the following requirements are fulfilled:

(F1) Every $a\in F$ is idempotent, that is, $a^2=a$;

(F2) For each $a\in F$ its range R_a determines a uniquely, that is, $R_a=R_b$ leads to $a=b$;

(F3) If $(p\colon p_a)=0$, then $p_a=0$ and $(p\colon q_a)=(q_a\colon p)=0$ for all $q\in P_0$†;

(F4) If $(p\colon p_a)=(p\colon q_a)\neq 0$, then $p_a=q_a$;

(F5) For any $a\in F$ and any non-zero $p\in P_0$ there exists the unique pure state $q\in P_0$ such that $q_a=0$ and $(p\colon p_a)+(p\colon q)=1$;

(F6) The mappings $e_p\colon P_0\to P_0$, where $p\neq 0$, defined by

$$e_pq=\begin{cases}p, & \text{if } q\not\perp p\\ 0, & \text{if } q\perp p,\end{cases}$$

all belong to F.

We shall now establish some facts about the orthogonality and the partial ordering in an abstract filter logic F. Obviously, the orthogonality and the partial ordering are defined in F in accordance with our previous results, that is:
two filters $a, b\in F$ are said to be *orthogonal* ($a\perp b$, in symbols), if $ab=ba=0$, where 0 denotes the zero mapping from P_0 to P_0.

We say that a is *stronger* than b (or that a *implies* b) and write $a\leqslant b$, if $ba=a$.

Proposition 5.5. For $a, b\in F$ the following statements are equivalent:
(i) $a\leqslant b$;
(ii) For every $p\in P_0$ satisfying $(p\colon p_a)=1$ we have also $(p\colon p_b)=1$;
(iii) $R_a\subseteq R_b$.

Proof. Let us assume (i), and let $(p\colon p_a)=1$, i.e. $p=p_a$. As $a\leqslant b$ means, by definition, that $ba=a$, we have $p_b=(p_a)_b=p_b$, hence $(p\colon p_b)=(p\colon p_a)=1$, which proves (ii).

The next implication, (ii)$\Rightarrow$(iii), is obvious, since for an arbitrary $x\in F$ we have $R_x=\{p\in P_0\colon (p\colon p_x)=1\}\cup\{0\}$ (R_x being, as before, the range of x).

Finally, assume (iii) and then prove (i). But this is trivial, since $p_a\in R_a\subseteq R_b$ implies $p_a=(p_a)_b=p_{ba}$ (all p), which means that $a=ba$ i.e. $a\leqslant b$, as desired.

Proposition 5.6. For $a, b\in F$ the following conditions are equivalent:
(i) $a\perp b$;
(ii) For every $p\in P_0$ for which $(p\colon p_a)=1$, we have $p_b=0$;
(iii) $R_a\perp R_b$.

Proof. Let us assume (i), and let $(p\colon p_a)=1$, i.e. $p=p_a$. As $a\perp b$ leads, by definition, to $ba=0$, we have $p_a=(p_a)_b=p_{ba}=0$, which proves (ii).

The next implication, (ii)$\Rightarrow$(iii), is obvious, as for every $x\in F$ we have $R_x=\{p\in P_0\colon (p\colon p_x)=1\}\cup\{0\}$ and $R_x^{\perp}=\{p\in P_0\colon p_x=0\}$.

† We write, as usual, p_a instead of ap.

Indeed, to prove the last equality, note that $p \in R_x^{\perp}$ implies $(p: p_x) = 0$; hence $p_x = 0$ by (F3), and conversely, $p_x = 0$ leads by (F3) to $(p: q_x) = (q_x: p) = 0$ for all $q \in P_0$, which means that $p \in R_x^{\perp}$.

Finally, assume (iii) and then prove (i). This is trivial, since $R_a \perp R_b$ implies $R_a \subseteq R_b^{\perp}$ and $R_b \subseteq R_a^{\perp}$, and therefore $p_a \in R_a$ (all p) leads to $(p_a)_b = 0$; similarly, $p_b \in R_b$ (all p) implies $(p_b)_a = 0$, which proves that $ab = ba = 0$, that is, $a \perp b$ as claimed.

Proposition 5.7. For every $a \in F$ we have $R_a^{\perp\perp} = R_a$ and $K_a^{\perp\perp} = K_a$, where K_a, the so-called kernel of a, is defined as the set $\{p \in P_0: p_a = 0\}$. Moreover, $K_a^{\perp} = R_a$.

Proof. As we always have $R_a \subseteq R_a^{\perp\perp}$, it remains to be shown that $R_a^{\perp\perp} \subseteq R_a$. Thus, let us assume that $p \perp R_a^{\perp}$, i.e. that $p \perp q$ for all $q \in P_0$ with $q_a = 0$. We shall show that $p \in R_a$. By (F5) there exists a $q \in K_a$ such that $(p: p_a) + (p: q) = 1$; hence $p = p_a$, as $(p: q) = 0$, and we thus have $p \in R_a$, which proves the inclusion.

Obviously, the equality $R_a^{\perp\perp} = R_a$ implies $K_a^{\perp} = (R_a^{\perp})^{\perp} = R_a$. Finally, $K_a^{\perp\perp} = R_a^{\perp\perp\perp} = R_a^{\perp} = K_a$, and the proposition is proved.

As an immediate consequence of the proposition above we obtain

Corollary 5.8. $R_a \subseteq R_b$ if and only if $K_a \supseteq K_b$. In particular, K_a determines the filter a uniquely.

Note now that every filter $a \in F$ may be identified with the function $f_a: P \to [0, 1]$ (where $P = P_0 \backslash \{0\}$) defined by $f_a(p) = (p: p_a)$, $p \in P$, since, having assumed $(p: p_a) = (p: p_b)$ for all $p \in P$, one readily deduces that $p_a = p_b$ whenever, for example, $p_b \neq 0$, and $p_a = p_b = 0$, provided $p_b = 0$, hence $a = b$. Moreover, one easily checks that the pair $(\{f_a\}_{a \in F}, P)$ satisfies all the axioms required for a standard event–phase-space structure. Furthermore, we find F to be orthoisomorphic with $\{f_a\}_{a \in F}$ (see propositions 5.5 and 5.6).

Summarising the results which we have obtained in this section, one can write

Theorem 5.9. Given a standard event–phase-space structure (L, P), there exists a transition probability space $(P_0, (:))$ such that L is orthoisomorphic to some logic of filters acting on P_0.

Conversely, for any logic F of filters acting on a transition probability space $(P_0, (:))$ there exists a standard event–phase-space structure (L, P), where P actually coincides with the set $P_0 \backslash \{0\}$, such that F is orthoisomorphic with L.

Remark. If we impose an additional restriction on the transition probability between p and q_a, namely (see Guz 1979)

(F7) $(p: q_a) = (p: p_a)(p_a: q_a)$ and $(q_a: p) = (q_a: p_a)(p_a: p)$ for all $p, q \in P_0$,

then some postulates among (F1)–(F6), being physically not evident (e.g. (F4) and a part of (F3)) become superfluous, as they are consequences of (F7) (see Guz 1979).

It should be noted at this moment that the first part of (F7) has been assumed as a postulate by Deliyannis (1976) for the case of the symmetric transition probability. It tells us (see also Deliyannis 1976) that the probability $(p: q_a)$ of passing from a pure state p to a pure state $p' = q_a$, in which the event a occurs with certainty, is the product (independence!) of the probability $(p: p_a)$ of the transition from p to p_a and the probability $(p_a: q_a)$ of the subsequent transition from p_a to q_a. In a general case of a non-symmetric transition probability, it is reasonable to adjoin to Deliyannis' assumption also the 'left-handed' counterpart, whose physical content is similar.

For the reason mentioned above, we will refer to (F7) as to the 'independence postulate' (see also Guz 1979). Note that (F7) is obviously satisfied in the conventional models of classical and quantum mechanics (see Deliyannis 1976).

Note, finally, that if we assume also the 'left-handed' counterpart of the first half of (F3), i.e. that $(p_a : p) = 0$ implies $p_a = 0$, one can then prove by using (F7) the following fact (see Guz 1979):

$$R_a \subseteq R_b \text{ iff } a = ba = ab.$$

Note that (F2) follows as a corollary of this result.

6. The representation theorem

Assume, in this section, the validity of axioms (B1)–(B4) for an event–phase-space structure (L, P). Additionally, assume also the postulate

(B5) If $p_a = 0$ and $b \perp a$, then also $(p^b)_a = 0$;

which is of technical significance for us.

In other words, the axiom (B5) tells us that $F_b(K_a) \subseteq K_a$, provided $b \perp a$. Note that when L possesses an orthocomplementation $' : L \to L$, by using which the orthogonality in L is defined by: $a \perp b$ iff $a \leq b'$, then (B5) is satisfied automatically. Indeed, suppose that $p_a = 0$ and $b \perp a$ for some $p \in P_0$ and a, $b \in L_0$ (note that one can assume, without loss of generality, that a and b are non-zero, and thus a, $b \in L$). Then $a(p) = (p : p_a) = 0$; hence $a \perp s(p)$, and therefore $a \perp s(p) \vee b$, since $s(p) \leq a'$ and $b \leq a'$ implies also $s(p) \vee b \leq a'$. But $s(p^b) = s(p) \vee b - b \leq s(p) \vee b \perp a$ implies $s(p^b) \perp a$; hence $(p^b : (p^b)_a) = a(p^b) = 0$, which leads to $(p^b)_a = 0$, as claimed.

Let us now consider the embedding $L_0 \to \tilde{L}_0$ described in § 3, where in place of L we now take $L_0 = L \cup \{0\}$. $\tilde{L}_0$ has, obviously, all the properties of $\tilde{L}$, and thus $\tilde{L}_0$ becomes a complete orthocomplemented lattice (with respect to the set inclusion) with the orthocomplementation given by $M \to M^\perp$ $(M \in \tilde{L}_0)$, and the orthoinjection of L_0 into $\tilde{L}_0$ is realised by the map $a \to a^{\perp\perp} = \{a\}^\Delta$.

Theorem 6.1. Suppose (L, P) is an event–phase-space structure satisfying axioms (B1)–(B5). Then $\tilde{L}_0$ is atomic and orthomodular, and satisfies the covering law.

Proof. The atomicity of L_0 is almost obvious, as it follows directly from the atomicity of L_0 (compare Guz 1978a, § 5). To prove the orthomodularity of L_0, we will need some lemmas.

Lemma 6.2. If for three events a, b, $e \in L_0$, where e is an atom, we have $a \perp b$ and $e \perp b$, then we also have $a \vee e \perp b$.

Proof. Let us suppose that a, $e \perp b$, where e is an atom. One can assume, without loss of generality, that $e \not\leq a$. Then by lemma 4.6 $e \vee a = f + a$, where f is an atom (orthogonal to a) chosen according to the prescription of the second half of the axiom (B3'), that is, $f = a(p^a)$, where $p = s^{-1}(e)$ (see (B3'')). Since $s(p) = e \perp b$, we have $(p : p_b) = b(p) = 0$, and hence $p_b = 0$ (see theorem 5.1); since also $a \perp b$, we have $(p^a)_b = 0$ by (B5), and hence $b(p^a)=(p^a:(p^a)_b)=0$ (see theorem 5.1), which leads to $b\perp s(p^a)=f$. But $b\perp a$, $b\perp f$ and $a\perp f$ imply, by lemma 4.2, $b\perp a+f=a\vee e$, as claimed.

Lemma 6.3. Let M be a non-empty subset of L_0; then
(i) $a\in M^{\perp\perp}$ and $b\leq a$ imply $b\in M^{\perp\perp}$;
(ii) If a, $e\in M^{\perp\perp}$, where e is an atom, then $a\vee e\in M^{\perp\perp}$.

Proof. $a\in M^{\perp\perp}$ means that $a\perp c$ for all $c\in M^{\perp}$, and therefore for every $b\leq a$ we also have $b\perp c$ for all $c\in M^{\perp}$, that is, $b\in M^{\perp\perp}$.

Suppose now that a, $e\in M^{\perp\perp}$, where e is an atom. Then, obviously, a, $e\perp c$ for all $c\in M^{\perp}$; hence also $a\vee e\perp c$ (all $c\in M^{\perp}$) by lemma 6.2, i.e. $a\vee e\in M^{\perp}$ as claimed.

Now let $M\in\tilde{L}_0$, $M\neq\{0\}$, and let B be a subset of M consisting of pairwise orthogonal atoms. Note that by virtue of the orthomodularity and Zorn's lemma, B can be extended to a maximal such set.

Any maximal subset $B\subseteq M$, consisting of pairwise orthogonal atoms, will be called an *orthobasis* of M (Bugajska and Bugajski 1973c).

Lemma 6.4. For any orthobasis B of $M(M\in\tilde{L}_0, M\neq\{0\})$ one has $B^{\perp\perp}=M$.

Proof. Let us suppose, to the contrary, that $B^{\perp\perp}\subsetneq M$, i.e. that there is $a\in M$ such that $a\not\perp b$ for some $b\in B^{\perp}$. Since $a\not\perp b$, there exists by (EPS 1) and (EPS 0) (see § 2) a pure state $p\in P$ such that $a(p)=1$ and $b(p)>0$; hence $s(p)\leq a$ and $s(p)\not\perp b$.

One can easily prove that for every $p\in P$ the function $f\to f(p)$, where $f\in B$, takes non-zero values only for at most a countable subset of B, say $\{f_1, f_2, \ldots\}$.

Further, $b\in B^{\perp}$ implies $b\perp f_i$ for all $i=1, 2, \ldots$; hence $b\perp\bigvee_i f_i$ by lemma 4.2, and therefore $s(p)\not\leq\bigvee_i f_i$, as otherwise we would have $s(p)\perp b$, a contradiction. Hence $s(p)\vee\bigvee_i f_i>\bigvee_i f_i$, and therefore, by the orthomodularity and atomicity, there exists an atom $f\leq s(p)\vee\bigvee_i f_i-\bigvee_i f_i$.

Since for all $g\in B\backslash\{f_1, f_2, \ldots\}$ we have $g(p)=0$, which leads to $s(p)\perp g$, and since also $\bigvee_i f_i\perp g$ (by lemma 4.2), we have $s(p)\vee\bigvee_i f_i\perp g$ by lemma 6.2, which leads to $f\perp g$ (for all $g\in B\backslash\{f_1, f_2, \ldots\}$). Therefore, we have $f\perp B$. But $f\in M$ by lemma 6.3, since all $f_i\in M=M^{\perp\perp}$, and hence $\bigvee_i f_i=\Sigma_i f_i\in M$ by lemma 4.2, and $s(p)\in M$, as $s(p)\leq a\in M$ (see lemma 6.3). We thus arrive at a contradiction with the maximality of B (see the definition of an orthobasis), and the lemma is therefore proved.

Now, we come back to the proof of the theorem, and to prove the orthomodularity we will follow the arguments of Bugajska and Bugajski (1973c). Let $M_1\subsetneq M_2$, M_1, $M_2\in\tilde{L}_0$, and let B_1 be an orthobasis of M_1 (the existence of at least one orthobasis for every $M\in\tilde{L}_0$ follows from Zorn's lemma, as we remarked before). One can then extend B_1 to some orthobasis B_2 of M_2, and obviously $B_1\neq B_2$, as otherwise $M_1=B_1^{\perp\perp}=B_2^{\perp\perp}=M_2$, which contradicts our assumption. Let $B=B_2\backslash B_1$ and $M=B^{\perp\perp}$. We then have, obviously, $M\perp M_1$, since $B\perp B_1$, and moreover

$$M_1\vee M=(M_1\cup M)^{\perp\perp}=(M_1^{\perp}\cap M^{\perp})^{\perp}=(B_1^{\perp}\cap B^{\perp})^{\perp}=(B_1\cup B)^{\perp\perp}=B_2^{\perp\perp}=M_2,$$

which proves the orthomodularity of $\tilde{L}_0$.

There yet remains to be shown the validity of the covering law in $\tilde{L}_0$. It will be sufficient to prove (see Guz 1978b) that statement (2) of theorem 4.7 holds for $\tilde{L}_0$, which, in turn, follows easily from the validity of (2) in L_0. In fact, let $\{0, e\}\subseteq(\{0, e_1\}\cup\{0, e_2\}\cup\{0, e_3\})^{\perp\perp}$, where $\{0, e\}\neq\{0, e_3\}$ and let $\{0, e\}\not\subseteq(\{0, e_1\}\cup\{0, e_2\})^{\perp\perp}$

(the atoms of $\tilde{L}_0$ are the subsets of the form $\{0, e\}$, where e are atoms in L_0). This, as may easily be seen, is equivalent to the assumption that $e \leqslant e_1 \vee e_2 \vee e_3$, $e \neq e_3$, $e \not\leqslant e_1 \vee e_2$, which implies by theorem 4.7(2) the existence of an atom $f \in L_0$ such that $f \leqslant e \vee e_3$ and $f \leqslant e_1 \vee e_2$, which may equivalently be written as $\{0, f\} \subseteq (\{0, e\} \cup \{0, e_3\})^{\perp\perp}$ and $\{0, f\} \subseteq (\{0, e_1\} \cup \{0, e_2\})^{\perp\perp}$. This shows that statement (2) of theorem 4.7 holds in $\tilde{L}_0$, which is equivalent to the validity of the covering law in $\tilde{L}_0$ (see Guz 1978b). Thus the proof of the theorem is complete.

Summarising, we find by the result of theorem 6.1 the conditions (B1)–(B5) imposed on an event–phase-space structure (L, P) to be sufficient to imply the well-known Piron–MacLaren representation theorem for $\tilde{L}_0$ (see, e.g., Piron 1964, MacLaren 1964, Varadarajan 1968, Maeda and Maeda 1970), and therefore for L_0 also, provided we assume $\tilde{L}_0$ to be irreducible and of dimension greater than three.

Note that the irreducibility of $\tilde{L}_0$ is not a severe restriction, as otherwise any irreducible part of $\tilde{L}_0$ may be taken in place of the whole $\tilde{L}_0$. Furthermore, the irreducibility of $\tilde{L}_0$ can also easily be understood from the physical viewpoint, as it can be formulated as the so-called 'superposition principle' (Guz 1978b):

$\tilde{L}_0$ is irreducible if and only if the set P of non-zero pure states possesses the following property, called the superposition principle: for any pair p, q of distinct pure states from P there exists a third pure state $r \in P$, $r \neq p, q$, called the superposition of p and q, such that $r \in \{p, q\}^{\perp\perp}$.

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